An Index of Littoral Zone Complexity and Its Measurement[1]

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The littoral zone is important in lake ecosystems. It affects physical, chemical, and biological processes of the whole lake. We first describe patterns in the shoreline lengths of a lake drawn from maps of different scales. Second, we show that these patterns or "measurement laws" hold for many different lakes. We then use "fractal" measuring theory to provide a unifying explanation of these empirical results. Fractal theory also makes new predictions about the statistical properties of groups of lakes. Patterns in our data were consistent with these predictions. Finally, we discuss how physical and geomorphological processes can give rise to fractal patterns in the landscape.

Key words: fractal, geomorphology, littoral, morpho-edaphic index, shoreline development index

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La zone littorale est importante dans les écosystèmes lacustres. Elle influe sur les processus physiques, chimiques et biologiques du lac entier. Nous décrivons en premier lieu la configuration de longueurs de lignes de rivage d'un lac dessinées à partir de cartes à échelles différentes. Nous démontrons en second lieu que ces configurations ou "lois de mesures" sont valables pour plusieurs lacs. Comme explication unifiant ces résultats empiriques, nous faisons appel à la théorie de la mesure des «fractures». Cette théorie permet aussi de nouvelles prédictions quant aux propriétés statistiques de groupes de lacs. Nos données s'accordent avec ces prédictions. Finalement, nous examinons la façon dont ces processus physiques et géomorphologiques produisent des fractures dans le paysage.

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THIS paper demonstrates how two physical properties of lakes, shoreline length and littoral area, can be accurately characterized. The results are used to predict how shoreline length changes with lake area. These relationships can also predict the distribution of lakes of different size-classes. Implications for the biological processes in the lake are discussed.

Shoreline and littoral properties are known to have important predictive value in biological relationships (Rigler 1973; Barsdate et al. 1974). Simple, empirical indices of littoral extent have been used to classify diverse lakes and to evaluate the importance of their littoral areas. One of the oldest morphometric indices is the shoreline development index, defined as

$$(1)\quad D_s = L/\sqrt{4\pi A}$$

where L is the measured shoreline length, and A is the lake area. A circle gives an index of 1; larger values should indicate more convoluted figures.

The development index has been used to relate nutrient loading in a lake to the length of its shoreline (Seppanen 1972; quoted in Dillon and Rigler 1975). It has also been used in statistical studies of fish community structure (Johnson et al. 1977). However, there is a paradox associated with the calculation of the index. While the total area of the lake can be measured fairly accurately, the measurement of circumference is more arbitrary. Measured lengths are dependent on either the detail of the map or the accuracy of the measuring instrument used. Shoreline lengths that we have determined by remeasuring detailed maps may be as much as 100% greater than published values. We need to know how finer or more detailed scales of measurement change overall length estimates.

The relationship between length and scale of measurement was first explored by Richardson (1961) for curves such as coastlines. A theoretical basis for his findings was provided by Mandelbrot (1967). The surprising result of these investigations was that the measured length of the curve did not reach an asymptotic, maximum value, but continued to increase with finer and finer scales of measurement. Furthermore, the relationship between measured length and the scale of measurement was allometric. This implies that these curves do not have a well-defined length. In practice, it means that

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TABLE 1. The latitude, longitude, and the scale of the map from which the lake outline was digitized, for the lakes used in this study.

Lake	Latitude	Longitude	Map scales used
Big Duck Lake	44°51′	78°57′	1:50 000
Bob Lake	44°55′	78°47′	1:50 000
Crotchet Lake	44°57′	78°56′	1:50 000
Gull Lake	44°51′	78°47′	1:50 000
			1:125 000
Lake Kawagama	45°18′	78°45′	1:50 000
			1:125 000
			1:500 000
Muskoka Bay	44°56′	79°24′	1:25 000
			1:500 000
Smudge Lake	44°49′	78°58′	1:50 000
Thrasher Lake	44°55′	78°59′	1:50 000

length determinations for different lakes will only be comparable if performed with the same level of detail, that is using the same scale of measurement. The length of a shoreline is then a scale-dependent quantity. This result is unsatisfying in that we cannot determine indices such as shoreline development independently of some arbitrary scale of measurement.

Mandelbrot (1977) explores the properties of highly convoluted shapes, of which river courses and coastlines are but two examples. He proposes an index which can be used to measure their degree of convolution. This index is called the fractal dimension. The fractal dimension can be determined empirically whenever there is an allometric relationship between the scale of measurement and the length measured at that scale. For a smooth curve, the fractal dimension D is 1; for more convoluted curves, D is between 1 and 2. It is determined as

$$(2)\quad D = 1 - m$$

where m is the slope of the allometric relationship:

$$(3)\quad \log[L(s)] = m \times \log[s] + a$$

where $L(s)$ is the length measured at a scale s.

Mandelbrot (1977) also predicts that lake area A and shoreline length L will be related for lakes with the same fractal dimensions D by the equation:

$$(4)\quad c_1 = L^{1/D}/\sqrt{4\pi A};\ \text{or}\ \ln(L) = c_2 + (D/2)\ln(A).$$

The c's are constant terms. Note that when $D > 1$, shoreline development is a biased measure of lake shoreline complexity. It will increase with lake size even for lakes with the same shape or shoreline complexity.

A third "fractal" law, known as Korcak's law, is noted by Mandelbrot (1977). If $F(A)$ gives the fraction of lakes in a region with area greater than A, lakes with fractal shoreline dimension D should satisfy the equation:

$$(5)\quad F(A) = k\,A^{-D/2}.$$

So there are several ways to test for fractal patterns in lake structure. We tested the applicability of the fractal measuring concept to Ontario lakes by investigating the relationship between shoreline complexity and littoral zone structure.

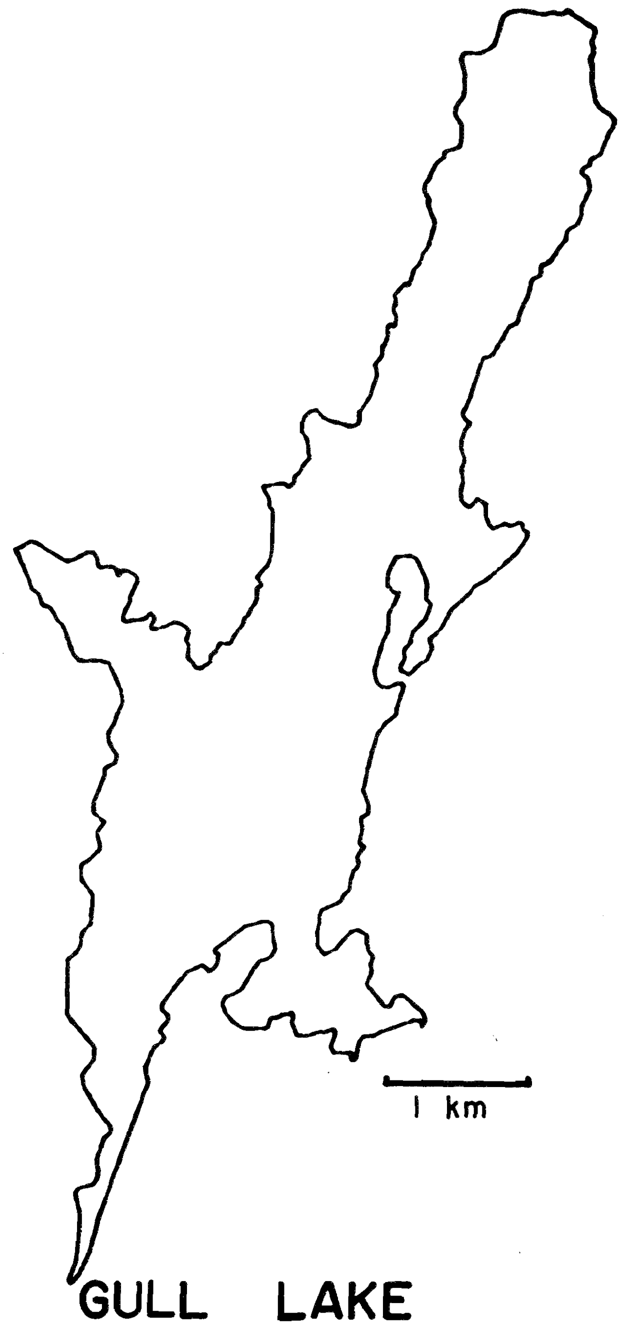

FIG. 1. Gull Lake as drawn by computer from digitized shoreline coordinates.

Materials and Methods

Maps at several different scales (1:25 000–1:500 000) were obtained for Canadian Shield lakes in the Muskoka and Haliburton regions of Ontario (Table 1). Lake shorelines were digitized on a Ruscom Co-ordinate Digitizer and stored as a series of x–y coordinates. The x–y coordinates fell within 0.5 mm of the original map outlines. The x–y coordinates

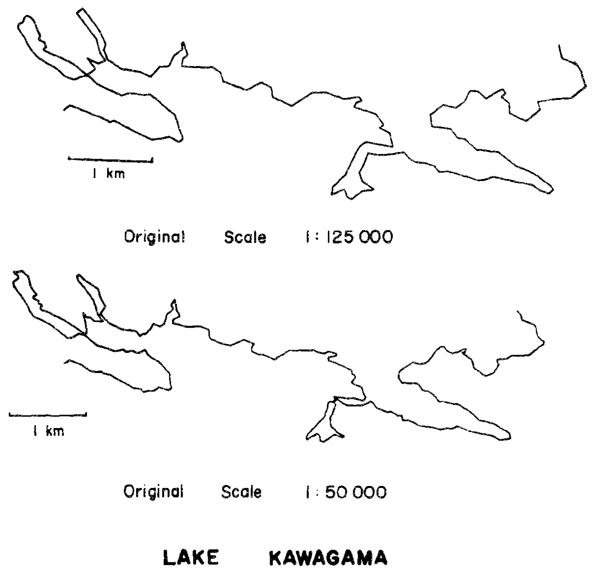

FIG. 2. Two computer-drawn outlines of a section of Lake Kawagama. Below each figure is the scale of the original map from which each shoreline was digitized.

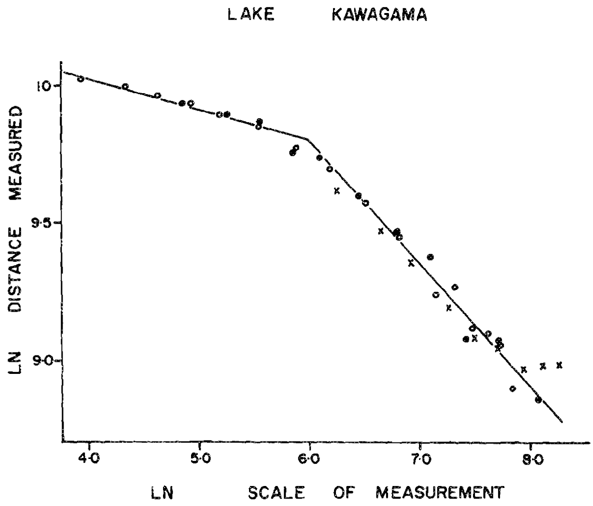

FIG. 3. The natural logarithm of the measured shoreline length (metres) is plotted against the natural logarithm of the scale of measurement (metres) for Lake Kawagama. Three different digitized representations of the same length of shoreline were produced. The x's show results from using a 1:500 000 map, the closed circles a 1:125 000 map, and the open circles a 1:50 000 map.

were joined to represent the digitized shoreline (Fig. 1). To measure a lake at a given scale s, a chain of connected line segments, each of fixed length s, was superimposed on the computer map so that: (a) the endpoints of each segment fell on the digitized shoreline; (b) shoreline path between endpoints was minimal. The measured length was then the number of line segments used times the length, s, of each segment. A correction was made to the length of the final segment so that the endpoints met. The measured length was affected by the starting point, so the procedure was replicated several times, using different starting points. This procedure is analogous to overlaying a chain with links of length s on the digitized shoreline, the ends of each link touching the shoreline, and then measuring the length of chain used.

Lake lengths were measured at scales, s, ranging from the smallest compatible with map detail to approximately one fifth of the length of the digitized shoreline. For certain lakes, several different maps were used. These maps had been drawn at scales ranging from 1:25 000 to 1:500 000. Digitized shorelines were obtained from each map. This additional information enabled us to evaluate the effect of map detail on the measuring procedure.

We tested the relationship (eqn. 4) between shoreline length and lake area by using a second data set, morphometric data on 21 lakes from the Lake Ecosystem Working Group. These lakes cover a somewhat wider region than the eight digitized lakes. As the shoreline length determinations were all taken from maps drawn at the same scale, we felt that these lengths were comparable.

To test the validity of Korcak's law, a third lake data set taken from Cox (1978) was used. Numbers of lakes with areas between 1–9 ha and 10–99 ha (Cox's size-classes V andVI) in several watersheds covering the study area (HF, HH) were used to fit equation 5.

Levels of detail on the above maps were not adequate to determine littoral zone areas, so an aerial photograph of Christie Lake (S. Ont.) with a scale of 1:7700 was digitized and measured as before. A series of zones was defined. Each zone consisted of all points within a fixed distance of shore. The areas of these zones were determined by planimetry. This constituted our fourth data set.

Results

Measurement Laws of Shoreline Lengths

Figure 2 shows the digitized outlines of a portion of Lake Kawagama taken from maps with scales 1:125 000 and 1:50 000, respectively. Figure 3 shows the results of the measuring procedure on Lake Kawagama for three different digitized shorelines. Intuitively, one would expect that the measured length would increase as the scale of measurement decreases. This is verified by the data, as there is a significant negative correlation between the measured shoreline length and the scale of measurement ($r^2 = 0.85$). If we tentatively entertain the hypothesis that lake shorelines follow fractal laws, we can make two types of predictions. We may find that we can extrapolate from large-scale to fine-scale measurements. Secondly we may find that lakes in the same geological province are similar in their shoreline structure. They should have the same fractal dimensions.

From Fig. 3, we can see that shoreline length is not described by a simple allometric relationship. While the measurement data for the largest scale map is linear within its range, the data for both the 1:50 000 and 1:125 00 scale maps show a significant lack of fit to a single linear model. This means that we cannot extrapolate to finer scales in order to predict fine-scale shoreline lengths. However, our second prediction appears to hold as all of the lakes show a similar pattern (Table 2). This pattern involves a second linear or

TABLE 2. Large- and small-scale slopes, log breakpoint (metres) from best fitting two-line regressions for each lake.

Lake	Slopes		Breakpoint (m)
	Large scale	Small scale	
Big Duck Lake	−0.64	−0.13	255
Bob Lake	−0.44	−0.16	482
Crotchet Lake	−0.28	−0.12	178
Gull Lake	−0.27	−0.10	354
Lake Kawagama	−0.45	−0.12	354
Muskoka Bay	−0.44	−0.16	459
Smudge Lake	−0.55	−0.16	255
Thrasher Lake	−0.43	−0.16	482

fractal relationship valid at finer scales of measurement. We therefore propose a modification to our first prediction.

Suppose the shoreline of a lake is formed by at least two different geomorphological fractal processes. Each process modifies the shoreline, but at different scales. If we measure the shoreline at the scales at which the effects of only one process predominate, then we find a single fractal law. If we had two processes working at two different scales, then the resulting measurement curve would have two parts. These two line segments would be joined at the characteristic scale or breakpoint separating the two processes. If the same processes are operating on all of the lakes in our study area, then the line segments should all bend at the same scale.

To test this model, the data was partitioned into two sets, consisting of measurements at scales above or below a scale called the breakpoint. A separate line was fitted to each set. We repeated this procedure, varying the breakpoint. We recorded the breakpoint for which the regressions fit best. The results are shown in Table 2.

This analysis revealed several patterns. (A) The two-line model fit very well for each lake. (B) The slopes of the upper and lower line segments fell into two different and distinct regions. Within each region the slopes showed remarkable consistency. (C) Two processes were sufficient to explain all the measured relationships, and these processes were all separated at similar scales or breakpoints. For each lake the two-line model explains significantly more variance than the one-line model. The large-scale lines correspond to fractal dimensions clustering around a mean value ($x \pm 2$ SE) of 1.44 ± 0.09, while the dimensions at the fine scales cluster around 1.14 ± 0.02. The breakpoint has a mean value of 350 m ± 90 m.

Mapping Inaccuracies

We then checked to see if the relationships were a result of the mapping procedure used, rather than natural properties of the lakes themselves. This was why we measured Lake Kawagama shorelines from maps at three different scales. The same procedure was also applied to Gull Lake. The results for Lake Kawagama are shown on Fig. 3 and for Gull Lake on Fig. 4. Although the original scales of the maps differ greatly, there is a remarkable consistency in the results. Measurements taken from crude large-scale maps overlap the measurements taken from fine-scale maps. Therefore the results summarized in Table 2 are independent of mapping inaccuracies.

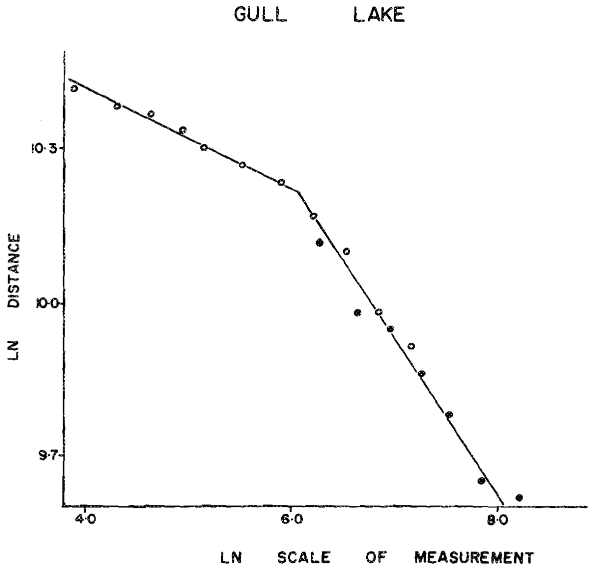

FIG. 4. The natural logarithm of the measured shoreline length (metres) is plotted against the natural logarithm of the scale of measurement (metres) for Gull Lake. Two different digitized representations of the lake were used. The closed circles show the results from using a 1:500 000 map, the open circles a 1:50 000 map.

Lake Area Versus Shoreline Length

The relation between shoreline length L (measured at a fixed scale) and lake area A was determined by a linear regression of $\ln(L)$ on $\ln(A)$. Equation 4 predicts that the slope of the fitted relationship will be half the fractal dimension D. For the 21 lakes in our second data set, the regression was highly significant; ($r^2 = 0.94, F_{(2,19)} = 268, P < 0.001$) the value of D was estimated to be 1.50 ± 0.18 (Fig. 5). This agrees with the large-scale shoreline dimension 1.44 ± 0.09 found earlier. When lakes from regions outside our study area (Parry Sound, Sudbury) were included, the estimated value of D rose to 1.59 ± 0.18 ($r^2 = 0.91$, $F_{(2,31)} = 305$, $P < 0.001$).

Distribution of Lake Areas

We also calculated the fractal dimension D from Korcak's law (eqn. 5) by using the relation:

$$(6) \quad \log_{10}\{(F(A) - F(10A))/(F(10A) - F(100A))\} = D/2.$$

When the value of A is taken to be 1 ha, the numerator and denominator above are just the numbers of lakes in size-classes V and VI in our third data set. The results are shown in Table 3. For the watersheds covering the central portion of our study area (2HF and 2HH), D was estimated to be 1.51, in agreement with our previous estimates. Other watersheds gave higher D values; the mean for five watersheds was 1.64 ± 0.11. These increased values behave like those found when equation 4 was applied to lakes from the north and west of our study area.

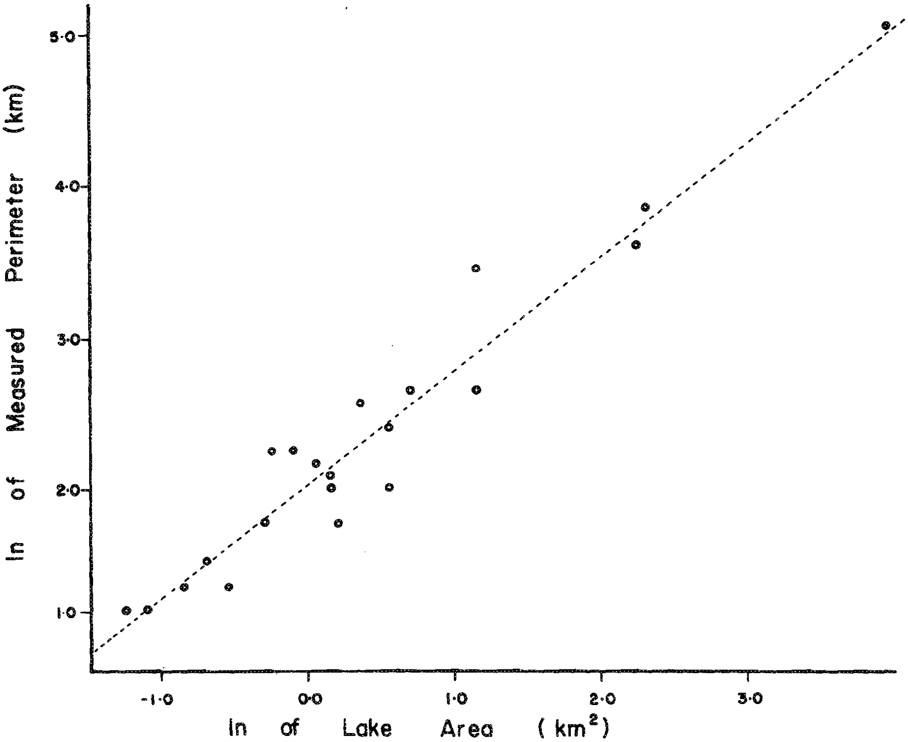

FIG. 5. The natural logarithm of lake area (km^2) is plotted against the natural logarithm of the measured perimeter of the lake (km) for 21 lakes in our study area. From the slope of this regression we are able to calculate an independent estimate of the average fractal dimension of the shoreline for lakes in this area.

TABLE 3. Areal distribution of lakes in watersheds: a test of Korcak's law.

Watershed[a]	LEWG lake group	Number of lakes in size-class V (10-99 ha)	Number of lakes in size-class VI (1–9 ha)	Estimate of D from eqn. 6
2HF+2HH	Kawartha, Muskoka	75	426	1.51
2KB+2KD	Algonquin	111	724	1.63
2DD+2CF	North Bay, Sudbury	150	1022	1.66

[a]Watersheds from Cox (1978). The correspondence between Cox's watersheds and the LEWG lake groupings is approximate.

FRACTAL SCALING OF NEARSHORE AREAS

Finally, we examined how littoral area relates to shoreline complexity. As detailed lake contour maps were not available at the scale at which we were working, we decided to test the simpler relationship of shoreline length to "nearshore" area. Specifically, we measured the area of zones within a given distance w of the shore. If the zone had smooth edges of length L then the area would approximately be equal to $L \times w$. However, we have shown that the length L depends on the scale at which the shoreline is measured according to equation 3, or setting the scale s equal to w:

(7) $L(w) = c \times w^{1-D}$

where c is a constant and D is the fractal dimension. The area as a function of the width w of the zone would then be given by

(8) $$A(w) = L(w) \times w$$
$$A(w) = c \times w^{1-D} \times w = c \times w^{2-D}.$$

For smooth curves, D is equal to 1. The constant c is then the length of the curve. The adjacent area is linearly proportional to the width of the zone. For fractal shorelines, D is greater than 1. Thus, in very convoluted lakes the shallowest parts of the littoral zone, the small bays and coves, make up a greater proportion of the zone than in less-convoluted lakes.

The heuristic argument leading to equation 8 gives a formula identical with that given by the Minkowski–Bouligand dimension law (Mandelbrot 1977). We wanted to see if these two relationships gave the same fractal dimension.

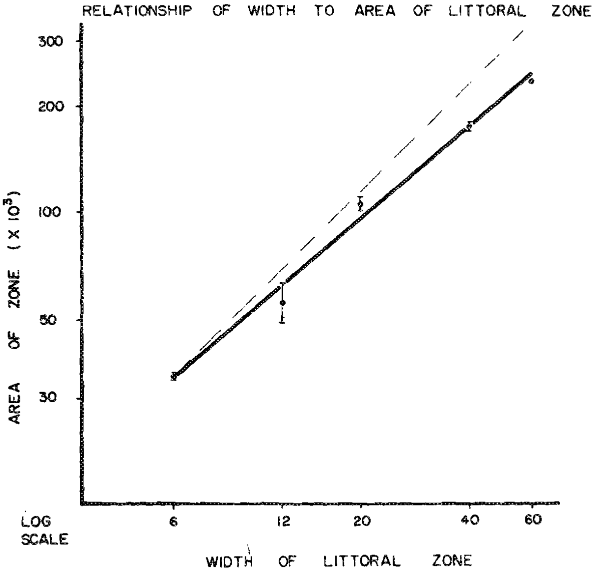

FIG. 6. The relationship between the width of the littoral zone (m) and its area (m^2) for Christie Lake. The broken line represents the expected relationship if the fractal dimension was equal to one. The open circles show the measured relationship (±2 SE).

Figure 6 shows the relationship between the width of the zone and the measured area for Christie Lake. The fractal shoreline dimension calculated from equation 5 was 1.149. When we regressed log(area) on log(width) we obtained a slope of 0.85 ($r^2 = 0.99$). This is significantly different from a slope of one ($F_{(1,26)} = 22.5, P < 0.001$). Using equation 8, we compute a fractal dimension of 1.15. Although this is but a single example, the excellent agreement between the values is highly suggestive.

Discussion

How do the fractal properties of lake shorelines contribute to our understanding of lake structure and biology? The shoreline development index is of "considerable interest because it reflects the potential for greater development of the littoral communities in proportion to the volume of the lake" (Wetzel 1977). But as Hutchinson (1957) points out, shoreline length "depends on the fineness of detail of the map and for this reason the values obtained on maps of different scale . . . may be somewhat different." Thus shoreline development "suffers from the same sources of uncertainty as measurements of length" (Hutchinson 1957). The determination of a fractal measurement law overcomes this difficulty by demonstrating a common basis for interpretation of shoreline measurements.

Lake Structure

Our data supported the hypothesis that lakes in the same region show similar fractal properties. Two fractal laws held at large and small scales separated by a breakpoint at a scale of about one third of a kilometre. How might this have come about?

Our lakes fall into Hutchinson's geomorphological type 26, in which glacial action has removed softer rock overburden and differentially deepened rock basins in zones of fracture and shatter belts (Hutchinson 1957). Ambrose (1964; quoted by Chorley 1969) suggests that ice scouring is "reactivating an ancient topography virtually intact." The fractal dimension ($D = 1.45$) found at large scales for shield lakes may reflect large-scale features, such as the distribution of fracture zones. The fractal dimension changes ($D = 1.15$) at a scale of measurement around 350 m. This should be the scale at which the glacial corrasive processes cancel each other out and erosional processes predominate. The dependence of glacial corrasion and erosional processes on preexisting geological features found over a wide area explains the consistency of the two dimensions and the breakpoint in the variety of lakes we measured. We predict that similar properties will be found for other irregular glacial scour lakes in different regions of the shield.

Three other predictions of fractal theory were tested. The shoreline length–lake area relationship of equation 4 and the lake area distribution law of equation 5 gave independent estimates of the shoreline dimension D of 1.50 and 1.51, respectively. These agree well with the large-scale D value of 1.44 found from the digitized lakes. The nearshore area relation of equation 8 independently estimated a D value of 1.15, identical with the small-scale shoreline D value. The excellent agreement of these independent dimension estimates supports the hypothesis that fractal laws govern the structure of the lakes in a region.

One consequence of the fractal structure of lakes is that the shoreline development index (eqn. 1) will be biased by lake size. If, as our data suggest, the ratio in equation 4 is constant, then shoreline development will increase with lake area. That is, if a lake were simply enlarged without changing its shape, the measured shoreline development would increase — in spite of the apparent correction for lake area which it incorporates.

In our second data set, shoreline development ranged from 3.1 to 11.7 and was significantly correlated ($P < 0.001$) with lake area. The ratio in equation 4, which represents a fractal measure of development, ranged from 0.91 to 1.28 and was uncorrelated with lake area. So fractal indices eliminate problems of scale because they are independent of measuring scale and lake size.

Community Structure

How can fractal morphometric laws help explain lake biology? We feel that fractal indices help quantify the role of the littoral zone. Most lake fish spawn in the littoral zone. The prey of many game fish are restricted to the littoral zone. In their classification of Ontario lakes, Johnson et al. (1977) attempted to explain the presence of certain fish species using limnological factors. In their analysis, the first discriminant function essentially separated deep, oligotrophic trout lakes from all others. However, the second discriminant function found the shoreline development index to be the most important of the physical factors. They interpreted this to be a measure of lake size. We suggest that this function might best be interpreted as a measure of nearshore habitat availability.

If this is true then why isn't the percent littoral area seen as a more important physical factor? As discussed above, the shallowest portions of the littoral zone, in particular weedy bays, may be disproportionately important to many species as foraging sites. Percent littoral area may include deeper areas of less importance, especially in clear oligotrophic lakes. The fractal dimension, on the other hand, is more concerned with the complexity of the shoreline, as it emphasizes the importance of shallow waters. It allows for the separation of lake size effects from those due to nearshore habitat availability.

Littoral Productivity

A reason for the importance of littoral zones, cited by Wetzel (1975), is the fast turnover of essential nutrients where decomposers are in close proximity to the primary producers. This increases the productivity of shallow waters. The fractal relationship between the area of a given zone and its mean width could help weight the contributions of shallow and deep-water regions in proportion to their net productivity. Such a weighting may be essential for a proper understanding of larger oligotrophic lakes. In lakes with low nutrient levels, extensive littoral zones may increase productivity (Adams and Olver 1977). Johnson et al. (1977) found that oligotrophic lakes with low Morpho-edaphic indices (MEI, Ryder 1965) were more likely to contain walleye (*Stizostedion vitreum vitreum*) if the lakes were large. They proposed that "larger lakes, with low overall MEI, can have higher MEI bays." Large lakes are seen as a patchwork of regions with different MEI values. The fractal dimension of the lake could be used to predict the distribution of these regions.

One major theme in studies of lake morphology involves the view that the littoral zone is a patchy environment of varying extent, which has a disproportionate influence on the lake as a whole. The fractal properties of lake shorelines and nearshore areas may eventually allow us to predict the distribution within a lake of high MEI areas and to correct indices of productivity for shoreline effects.

Extensions and Applications of Fractal Concepts

The fractal laws of basin morphology found above should extend to the topography of entire drainage basins. Erosional processes can produce fractal landforms (Leopold and Langbein 1962; Mandelbrot 1977). An allometric relationship exists between stream order and catchment area, stream length, maximum and mean discharge, etc. (Horton 1945; Leopold and Miller 1956; Hack 1957). A fractal measurement law of river length holds for all but the largest rivers (Richardson 1961). There should be a relationship between the fractal dimension of a stream course and the fractal dimension of its watershed (Mandelbrot 1977). Therefore, in regions where erosion has shaped the landforms, lakes formed by impoundment should show fractal shoreline laws predictable from the fractal properties of nearby drainage systems.

We propose then, that in regions where previous landforms largely determine lake outline, terrestrial contour maps could be used to test fractal theory. Contour lines in valleys would be measured to determine their fractal properties. These could be compared to the area between successive contours. In this way the fractal area law could be checked when zones of constant depth are used.

Because fractal laws can predict the distribution of areas of different habitats (lakes, bays, islands), they should be important in biogeographic models. By carefully documenting the statistical properties of landform structures, a first step can be taken towards quantifying habit complexity.

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